

Modified Chaplygin Gas with Variable G and Λ

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In this work, we have considered modified Chaplygin gas with variable G and Λ . The trivial solution describes decelerating phase to accelerating phase of the universe. The non-static with constant equation of state describes the inflationary solution. For static universe, G and Λ must be forms arbitrary and for static universe with constant equation of state G and Λ should be constant.

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I. INTRODUCTION

The Einstein field equation has two parameters - the gravitational constant G and the cosmological constant Λ . The Newtonian constant of gravitation G plays the role of a coupling constant between geometry and matter in the Einstein field equations. In an evolving Universe, it appears natural to look at this “constant” as a function of time. Numerous suggestions based on different arguments have been proposed in the past few decades in which G varies with time [1]. Dirac [2] proposed a theory with variable G motivated by the occurrence of large numbers discovered by Weyl, Eddington and Dirac himself. Many other extensions of Einstein’s theory with time dependent G have also been proposed in order to achieve a possible unification of gravitation and elementary particle physics or to incorporate Mach’s principle in general relativity [3].

From the point of view of incorporating particle physics into Einstein’s theory of gravitation, the simplest approach is to interpret the cosmological constant Λ in terms of quantum mechanics and the Physics of vacuum [4]. The Λ term has also been interpreted in terms of the Higgs scalar field [5]. Linde [6] proposed that Λ is a function of temperature and related it to the process of broken symmetries. Gaspirini [7] in this regard argues that Λ can also be interpreted as a measure of temperature of a vacuum which should decrease like the radiation temperature with cosmic acceleration. By considering the conservation of the energy-momentum tensor of matter and vacuum take together, many authors have invoked the idea of a decreasing vacuum energy and hence a varying cosmological constant Λ with cosmic expansion in the frame work of Einstein’s theory.

Λ as a function of time has also been considered by several authors in various variable G theories in different contexts [8]. Investigating the distance dependence of gravity under very general conditions, Wilkins [9] found that the gravity field at a distance r from a point mass has two components: one varying as r^{-2} , the other as r (Hookian field). The latter component is identifiable with the weak field limit of the Λ term in Einstein’s equation. His analysis allows one to consider both the gravity fields - the Hookian field, coupled to Λ and the Newtonian one coupled to G - on an equal footing. With this in view, several authors [10, 11] have proposed linking the variation of G with that of Λ in the framework of general relativity. This approach preserves conservation of the energy-momentum tensor of matter and leaves the form of the Einstein field equations unchanged. Though this approach is non-covariant, it is worth studying because it may be a limit of some higher dimensional fully covariant theory [10, 12].

Recent observations of the luminosity of type Ia Supernovae indicate [13, 14] an accelerated expansion of the Universe and lead to the search for a new type of matter which violates the strong energy condition i.e., $\rho + 3p < 0$. The matter content responsible for such a condition to be satisfied at a certain stage of evolution of the universe is referred to as a *dark energy*. There are different candidates to play the role of the dark energy. The type of dark energy represented by a scalar field is often called Quintessence. The simplest candidate for dark energy is Cosmological Constant Λ . In particular one can try another type of dark energy, the so-called Chaplygin gas which obeys an equation of state like [15] $p = -B/\rho$, ($B > 0$), where p and ρ are respectively the pressure and energy density. Subsequently the above equation was generalized to the form [16] $p = -B/\rho^n$, $0 \leq n \leq 1$. There are some works on modified Chaplygin Gas obeying equation of state [17, 18]

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$$p = A\rho - B/\rho^n, (A > 0) \quad (1)$$

At all stages it shows a mixture. This is described from radiation era to Λ CDM model.

II. EINSTEIN'S FIELD EQUATIONS

We consider the homogeneous and isotropic space-time given by FRW metric

$$ds^2 = dt^2 - a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \quad (2)$$

where k ($= 0, \pm 1$) is the curvature parameter.

The energy-momentum tensor for perfect fluid is

$$T_{ij} = (\rho + p)u_i u_j + p g_{ij} \quad (3)$$

where ρ and p are energy density and isotropic pressure respectively and $c = 1$.

The Einstein field equations with variable G and Λ is given by

$$R_{ij} - \frac{1}{2} R g_{ij} - \Lambda(t) g_{ij} = -8\pi G(t) T_{ij} \quad (4)$$

i.e., we have two equations as

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G(t)\rho}{3} + \frac{\Lambda(t)}{3} - \frac{k}{a^2} \quad (5)$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G(t)}{3}(\rho + 3p) + \frac{\Lambda(t)}{3} \quad (6)$$

From (5) and (6) we have

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) + \rho \frac{\dot{G}}{G} + \frac{\dot{\Lambda}}{8\pi G} = 0 \quad (7)$$

We assume the law of conservation of energy ($T_{;j}^{ij} = 0$) giving

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \quad (8)$$

From (7) and (8) we have

$$\dot{\Lambda} = -8\pi \dot{G} \rho \quad (9)$$

This implies $\dot{G} >$ or < 0 according as $\dot{\Lambda} <$ or > 0 i.e., G increases or decreases according to whether Λ decreases or increases.

Now for modified Chaplygin gas (1) we obtain

$$\rho = \left[\frac{B}{1+A} + \frac{C}{a^{3(1+A)(1+n)}} \right]^{\frac{1}{1+n}} \quad (10)$$

Now from equation of state (1), field equation (5) and conservation equation (8) we have

$$\frac{\dot{\rho}^2}{\rho^3} = 9 \left(1 + A - \frac{B}{\rho^{n+1}} \right)^2 \left(\frac{8\pi G}{3} + \frac{\Lambda}{3\rho} - \frac{k}{a^2\rho} \right) \quad (11)$$

Differentiating w.r.t t , we obtain

$$\dot{\rho} \left(\frac{2\ddot{\rho}}{\rho} - \frac{3\dot{\rho}^2}{\rho^2} \right) - \frac{3B(n+1)\dot{\rho}^3}{\rho^{n+3} \left(1 + A - \frac{B}{\rho^{n+1}} \right)} = 3\dot{\rho} \left(1 + A - \frac{B}{\rho^{n+1}} \right) \left[3 \left(1 + A - \frac{B}{\rho^{n+1}} \right) \left(\frac{k}{a^2} - \frac{\Lambda}{3} \right) - \frac{2k}{a^2} \right] \quad (12)$$

Now we consider $\dot{\rho} \neq 0$ (the case $\dot{\rho} = 0$ i.e., $\rho = \text{constant}$ is discussed later). Since the above differential equation of ρ is highly non-linear, so it cannot be solve analytically. Now for simplicity of calculation, we choose

$$3 \left(1 + A - \frac{B}{\rho^{n+1}} \right) \left[3 \left(1 + A - \frac{B}{\rho^{n+1}} \right) \left(\frac{k}{a^2} - \frac{\Lambda}{3} \right) - \frac{2k}{a^2} \right] = 0 \quad (13)$$

and

$$\left(\frac{2\ddot{\rho}}{\rho} - \frac{3\dot{\rho}^2}{\rho^2} \right) - \frac{3B(n+1)\dot{\rho}^2}{\rho^{n+3} \left(1 + A - \frac{B}{\rho^{n+1}} \right)} = 0 \quad (14)$$

From equation (13) we find the trivial solution of Λ i.e.,

$$\Lambda = \frac{(1+3A)k}{(1+A)a^2} - \frac{2kB}{C(1+A)^2} a^{3(1+A)(1+n)-2} \quad (15)$$

Also from equation (14) we obtain (after manipulation)

$$a^{\frac{3(1+A)}{2}} {}_2F_1 \left[\frac{1}{2(1+A)}, \frac{1}{2(1+n)}, 1 + \frac{1}{2(1+n)}, -\frac{Ba^{3(1+A)(1+n)}}{C(1+A)} \right] = \frac{\rho_0}{2} C^{\frac{1}{2(1+n)}} t \quad (16)$$

For $k > 0$, Λ decreases with t upto certain stage of the evolution of the universe and for $k < 0$, Λ increases with t after certain stage of the evolution of the universe (see figure 1) and for $k = 0$, we must have $\Lambda = 0$.

From (9) and (15) we have

$$4\pi G = \frac{\frac{(1+3A)k}{(1+A)a^3} + \frac{kB\{3(1+A)(1+n)-2\}}{C(1+A)^2} a^{3\{(1+A)(1+n)-1\}}}{\left[\frac{B}{1+A} + \frac{C}{a^{3(1+A)(1+n)}} \right]^{\frac{1}{1+n}}} \quad (17)$$

Note that here G_0 depends on the value of k i.e., $k = 0$ implies $G_0 = 0$. From figure 2 we see that G increases with the evolution of the universe for $k = 1$.

For early universe i.e., for $a \approx 0$, we get

$$4\pi G \approx \frac{(1+3A)k a^{1+3A}}{(1+A)C^{\frac{1}{1+n}}} + 4\pi G_0, \quad (G_0 > 0) \quad (18)$$

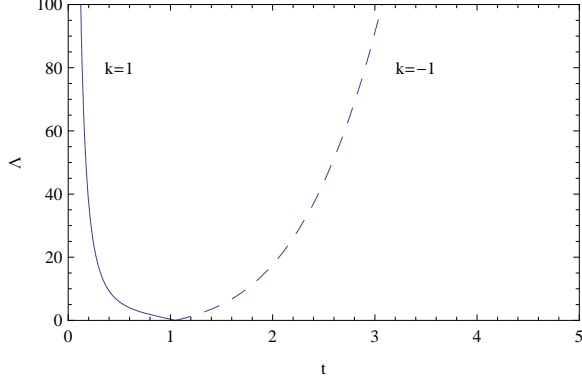


Fig.1

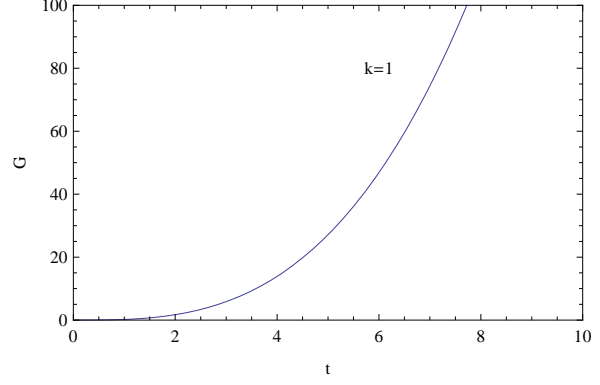


Fig.2

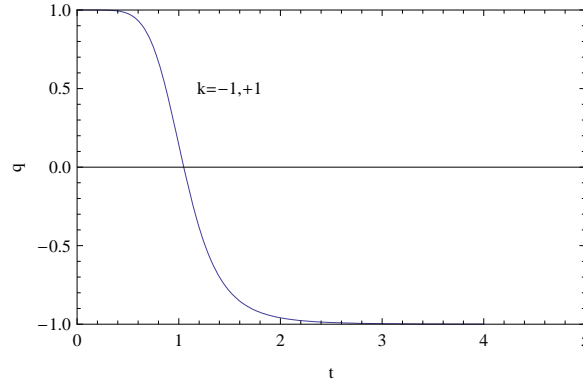


Fig.3

Fig. 1, 2 and 3 show the variations of Λ , G and q against t for $A = 1/3$, $B = 1$, $C = 1$, $n = 1/2$.

i.e., $G \rightarrow G_0$ as $a \rightarrow 0$.

Also for late universe i.e., for $a \approx \infty$, we have

$$4\pi G \approx \frac{kB}{C(1+A)^2} \left(\frac{1+A}{B} \right)^{\frac{1}{1+n}} a^{3(1+A)(1+n)-2} + 4\pi G_0 \quad (19)$$

i.e., $G \rightarrow \infty$ as $a \rightarrow \infty$.

The deceleration parameter has the expression

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = \frac{4\pi G(\rho + 3p) - \Lambda}{8\pi G\rho + \Lambda - 3ka^{-2}} \quad (20)$$

From figure 3 we see that q decreases from $+1$ to -1 for $A = 1/3$, $k = \pm 1$. This implies the universe has early deceleration and late acceleration. For $k = 0$, we have $\Lambda = 0$ and $G = 0$. So the field equations (5) and (6) yield $a = \text{constant}$, which is static solution.

The above discussions are valid for $\dot{\rho} \neq 0$ i.e., $\rho \neq \text{constant}$. Now we will discuss in the case $\dot{\rho} = 0$ i.e., $\rho = \text{constant} = \rho_0$ (say).

Now equation (8) reduces to

$$(\rho + p) \frac{\dot{a}}{a} = 0 \quad (21)$$

So we have possibilities: $\rho + p = 0$ or $\dot{a} = 0$.

(i) $\dot{a} \neq 0$ and $\rho + p = 0$: This implies $p = -\rho = -\rho_0 = p_0$ (say). From equation (1) we have $\rho_0 = \left(\frac{B}{1+A}\right)^{\frac{1}{1+n}}$. So from field equations (5) and (6), we get

$$a = \sqrt{\frac{k}{C_1}} \cosh(\sqrt{C_1} t)$$

From equation (9) we have

$$\Lambda = \Lambda_0 - 8\pi G \left(\frac{B}{1+A}\right)^{\frac{1}{1+n}}, \quad (\Lambda_0 = 3C_1) \quad (22)$$

There are no other equations, so Λ and G can not be calculated. We also see that in this case, G increases or decreases according as Λ decreases or increases. Λ and G are arbitrary functions of time in this case.

The field equations (5) and (6) and equation (22), we have

$$t = \sqrt{\frac{3}{\Lambda_0}} \log \left(\frac{\sqrt{\Lambda_0} a + \sqrt{\Lambda_0 a^2 - 3k}}{\sqrt{\Lambda_0} a_0 + \sqrt{\Lambda_0 a_0^2 - 3k}} \right)$$

i.e.,

$$a = a_0 \cosh\left(\sqrt{\frac{\Lambda_0}{3}} t\right) + \sqrt{a_0^2 - \frac{3k}{\Lambda_0}} \sinh\left(\sqrt{\frac{\Lambda_0}{3}} t\right) \quad (23)$$

For $k = 0$ we have $a = a_0 e^{\sqrt{\frac{\Lambda_0}{3}} t}$ which is the inflationary solution.

(ii) $\dot{a} = 0$ and $\rho + p \neq 0$: This implies $a = \text{constant} = a_0$ i.e., we have static universe. In this case, from field equations we have the values of G and Λ as

$$G = \frac{k}{4\pi a_0^2 \left[(1+A)\rho_0 - \frac{B}{\rho_0^n}\right]} = \text{constant} \quad (24)$$

and

$$\Lambda = \frac{k}{a_0^2} \left[\frac{(1+3A)\rho_0^{n+1} - 3B}{(1+A)\rho_0^{n+1} - B} \right] = \text{constant} \quad (25)$$

where $\rho_0 \neq \left(\frac{B}{1+A}\right)^{\frac{1}{n+1}}$.

(iii) $\dot{a} = 0$ and $\rho + p = 0$: This implies $a = \text{constant} = a_0$ i.e., we also have static universe and $p = -\rho = -\rho_0 = p_0$ (say). From equation (1) we have $\rho_0 = \left(\frac{B}{1+A}\right)^{\frac{1}{n+1}}$. From field equations, G and Λ satisfies:

$$\Lambda + 8\pi G \rho_0 = \frac{3k}{a_0^2} \quad (26)$$

and

$$\Lambda + 8\pi G\rho_0 = 0 \quad (27)$$

which are consistent only for $k = 0$. This implies

$$\Lambda = -8\pi G\rho_0 = 8\pi G \left(\frac{B}{1+A} \right)^{\frac{1}{n+1}} \quad (28)$$

The above relation (28) shows that G and Λ are arbitrary functions of time t .

III. CONCLUDING REMARKS

From the field equations and the conservation equation, an equation is obtained in ρ , a and Λ which suggests two trivial solutions. The trivial case with Λ is given in equation (15) leads to a model which starts from big bang with non-zero gravitational constant G_0 ($k \neq 0$) and has a positive constant deceleration parameter $q = 1$ and leads to accelerating universe ($q = -1$) with infinite gravitational constant G (see figure 3). For $k = 0$ we have $\Lambda = 0$ and $G = 0$ and $a = \text{constant}$, which is static solution. The model describes the evolution of the universe with early deceleration and late acceleration. For constant density ($\rho_0 \neq \left(\frac{B}{1+A} \right)^{\frac{1}{n+1}}$) universe, the cosmological constant Λ and gravitational constant G are arbitrary functions of time in which for static universe describes $k = 0$. But for static universe with non-constant density, G and Λ are constant where constant density $\rho_0 \neq \left(\frac{B}{1+A} \right)^{\frac{1}{n+1}}$.

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References:

- [1] P. S. Wesson (1978), *Cosmology and Geophysics* (Oxford : Oxford University Press); P. S. Wesson (1980), *Gravity, Particles and Astrophysics* (Dordrecht : Rieded).
- [2] P. A. M. Dirac, *Proc. R. Soc. A* **165** 119 (1938) ; **365** 19 (1979) ; **333** 403 (1973); The General Theory of Relativity (New York : Wiley) 1975.
- [3] F. Hoyle and J. V. Narlikar, *Proc. R. Soc. A* **282** 191 (1964); *Nature* **233** 41 (1971); C. Brans and R. H. Dicke, *Phys. Rev.* **124** 925 (1961).
- [4] Y. B. Zeldovich, *Sov. Phys.- JETP* **14** 1143 (1968); *Usp. Fiz. Nauk* **11** 384 (1968); P. J. E. Peebles and B. Ratna, *Astrophys. J.* **325** L17 (1988).
- [5] P. G. Bergmann, *Int. J. Theor. Phys.* **1** 25 (1968); R. V. Agoner, *Phys. Rev. D* **1** 3209 (1970).
- [6] A. D. Linde, *JETP Lett.* **19** 183 (1974).
- [7] M. Gasperini, *Phys. Lett.* **194B** 347 (1987); *Class. Quantum Grav.* **5** 521 (1988).
- [8] A. Banerjee, S. B. Dutta Chaudhuri and N. Banerjee, *Phys. Rev. D* **32** 3096 (1985); O. Bertolami *Nuovo Cimento* **93B** 36 (1986); *Fortschr. Phys.* **34** 829 (1986); Abdussattar and R. G. Vishwakarma, *Class. Quantum Grav.* **14** 945 (1997); Arbab I. Arbab, *Class. Quantum Grav.* **20** 93 (2003).
- [9] D. Wilkins, *Am. J. Phys.* **54** 726 (1986).
- [10] D. Kaligas, P. Wesson and C. W. F. Everitt, *Gen. Rel. Grav.* **24** 351 (1992).
- [11] A - M. M. Abdel Rahaman, *Gen. Rel. Grav.* **22** 655 (1990); M. S. Berman, *Gen. Rel. Grav.* **23** 465 (1991); A. Beesham, *Int. J. Theor. Phys.* **25** 1295 (1986).
- [12] P. S. Wesson, *Gen. Rel. Grav.* **16** 193 (1984).
- [13] N. A. Bachall, J. P. Ostriker, S. Perlmutter and P. J. Steinhardt, *Science* **284** 1481 (1999).
- [14] S. J. Perlmutter et al, *Astrophys. J.* **517** 565 (1999).
- [15] A. Kamenshchik, U. Moschella and V. Pasquier, *Phys. Lett. B* **511** 265 (2001); V. Gorini, A. Kamenshchik, U. Moschella and V. Pasquier, *gr-qc/0403062*.
- [16] V. Gorini, A. Kamenshchik and U. Moschella, *Phys. Rev. D* **67** 063509 (2003); U. Alam, V. Sahni , T. D.

- Saini and A.A. Starobinsky, *Mon. Not. Roy. Astron. Soc.* **344**, 1057 (2003); M. C. Bento, O. Bertolami and A. A. Sen, *Phys. Rev. D* **66** 043507 (2002).
- [17] H. B. Benaoum, *hep-th/0205140*.
- [18] U. Debnath, A. Banerjee and S. Chakraborty, *Class. Quantum Grav.* **21** 5609 (2004).